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AdS/CFT v.s. string loops

Gianluca Grignani

Dipartimento di Fisica and Sezione I.N.F.N., Università di Perugia
Via A. Pascoli I-06123, Perugia, Italia
E-mail: E-mail:grignani@pg.infn.it

Marta Orselli

NORDITA
Blegdamsvej 17, DK-2100 Copenhagen, Denmark
E-mail: E-mail:orselli@nbi.dk

Bojan Ramadanovic, Gordon W. Semenoff and Donovan Young

Department of Physics and Astronomy, University of British Columbia
6224 Agricultural Road, Vancouver, British Columbia V6T 1Z1 Canada
E-mail: bramadan@phas.ubc.ca, gordonws@phas.ubc.ca, dyoung@phas.ubc.ca

Abstract: The one string-loop correction to the energies of two impurity BMN states are computed using IIB light-cone string field theory with an improved 3-string vertex that has been proposed by Dobashi and Yoneya. As in previous published computations, the string vertices are truncated to the 2-impurity channel. The result is compared with the prediction from non-planar corrections in the BMN limit of $\mathcal{N}=4$ supersymmetric Yang-Mills theory. It is found to agree at leading order — one-loop in Yang-Mills theory — and is close but not quite in agreement at order two Yang-Mills loops. Furthermore, in addition to the leading 1/2 power in the t'Hooft coupling, which is generic in string field theory, and which we have previously argued cancels, we find that the 3/2 and 5/2 powers are also miraculously absent.

Keywords: AdS-CFT Correspondence, String Field Theory, Penrose limit and pp-wave background.

Contents

1. Introduction and conclusions

The AdS/CFT correspondence [1 − 3] has provided one explicit example of the long conjectured duality between gauge fields and strings. One of the most important testing grounds for this correspondence is string theory in the pp-wave geometry and its mapping to the BMN limit of Yang-Mills theory.

The pp-wave geometry is produced by taking the Penrose limit of $\mathrm{AdS}_5 \times \mathrm{S}^5$ [4, 5]. On that geometry, non-interacting IIB string theory is explicitly solvable and the complete spectrum of free strings can be found [6]. The corresponding BMN limit of $\mathcal{N} = 4$ super-Yang-Mills theory can be taken by identifying the appropriate operators [7] and taking a large quantum number limit. The planar limit of Yang-Mills theory corresponds to non-interacting strings and the planar spectrum of the Yang-Mills dilatation operator, which is dual to the string Hamiltonian, can be computed perturbatively [7]–[11]. As far as these computations have been done, the result shows beautiful agreement between planar Yang-Mills and non-interacting strings. This agreement has been extended to scenarios beyond the BMN limit [12, 13] and to the non-perturbative sector [14, 15] and has led to many promising insights.

One of those insights has been the recognition that the problem of computing dimensions of composite operators in $\mathcal{N} = 4$ super-Yang-Mills theory can be mapped onto integrable spin chains [16]–[18]. The string theory sigma model on $AdS_5 \times S^5$ also has an integrable structure [19] and much progress has been made to the point that a complete matching of the precise details of planar Yang-Mills and and non-interacting strings on the full $AdS_5 \times S^5$ background is a possibility that is sometimes contemplated [20]–[23].

However, in spite of this optimistic outlook, beyond the planar limit of Yang-Mills theory and non-interacting string theory, there has been very little success in checking the AdS/CFT correspondence, even in the BMN limit. For example, the Yang-Mills prediction for the string-loop corrections to energies of 2-impurity BMN states were computed early on [8, 9, 24, 25]. The gauge theory prediction for the BMN energy of a 2-impurity state is[1]

$$\Delta - J = 2\left(1+\frac{1}{2}\lambda' n^2 - \frac{1}{8}\lambda'^2 n^4 + \cdots\right) + \frac{g_2^2}{4\pi^2}\left(\frac{1}{12}+\frac{35}{32\pi^2 n^2}\right)\left(\lambda' - \frac{1}{2}\lambda'^2 n^2\right) + \cdots \quad (1.3)$$

Attempts to produce a result which matches this one using string theory have spawned a large literature [26]–[69] the best available published computation using light-cone string field theory is due to Gutjahr and Pankiewicz [65] (their eq. (4.17))

$$\begin{aligned}\frac{p^-}{\mu} &= 2\sqrt{1+\lambda' n^2} + \frac{g_2^2\lambda'}{4\pi^2}\times \qquad (1.4)\\ &\times\left(\left(\frac{1}{24}+\frac{65}{64\pi^2 n^2}\right) - \frac{3}{16\pi^2}\lambda'^{\frac{1}{2}} - \frac{n^2}{2}\left(\frac{1}{24}+\frac{89}{64\pi^2 n^2}\right)\lambda' + \frac{9n^2}{32\pi^2}\lambda'^{\frac{3}{2}} + \cdots\right).\end{aligned}$$

This computation, as did those which preceded it, uses an unjustified truncation of the string vertex to the 2-impurity channel. It clearly does not match the gauge theory result (1.3). The three-string vertex actually has an arbitrary pre-factor, the choice of which gives an arbitrary re-scaling of the entire expression in eq. (1.4). The pre-factor can thus be chosen so that either the first or the second term in the leading order λ' contribution agrees with gauge theory, but not both. There are other differences in the terms beyond the leading order.

On the other hand, in spite of its shortcomings, the formula in (1.4) has some remarkable features. The natural expansion parameter on the string side is $\lambda'^{\frac{1}{2}}$. In (1.4), the naive leading term that one would expect from power-counting, $\sim \lambda'^{\frac{1}{2}}$, is absent. It was argued that this is generally so in ref. [68]. The leading non-zero term, of order λ', has contributions of the same functional form in n as the gauge theory result, it is only the coefficients that are wrong. The bigger problem begins with the order $\lambda'^{\frac{3}{2}}$ term which is clearly absent in the gauge theory, where the expansion parameter is in integer powers of λ'. One might argue that such a fractional power is generated non-perturbatively, by re-summing logarithmic divergent diagrams for example, and it could appear in principle. This does happen elsewhere, for example in the expansion of the free energy of Yang-Mills theory at finite temperature. However, the gauge theory result seems to be free of infrared problems, this has been checked explicitly to at least order λ'^2, so it is difficult to see how a term of order $\lambda'^{\frac{3}{2}}$ could occur.

In the present paper, we will repeat the light-cone string field theory computation that led to (1.4), using the same truncation to the 2-impurity channel, and a modified form of the pp-wave background string vertex which was suggested by Dobashi and Yoneya in ref. [64]. Other details of the computation are identical to those in ref. [65] which led

[1]We remind the reader that the string light-cone momenta are related to Yang-Mills conformal dimension Δ and R-charge J as

$$p^- = \mu(\Delta - J)\,, \qquad p^+ = \frac{\Delta + J}{2\mu\sqrt{g_{YM}^2 N}\alpha'} \qquad (1.1)$$

where in the BMN limit $N, \Delta, J \to \infty$ so that (p^+, p^-) remain finite. Two convenient couplings are

$$\frac{1}{(\mu\alpha' p^+)^2} = \frac{g_{YM}^2 N}{J^2} \equiv \lambda'\,, \qquad 4\pi g_s\left(\mu\alpha' p^+\right)^2 = \frac{J^2}{N} \equiv g_2\,, \qquad N, J \to \infty \qquad (1.2)$$

λ' is proportional to the string tension. g_2 is the string coupling which weights the genus of the string world-sheet.

to (1.4).[2] Our result will be

$$\frac{p^-}{\mu} = 2\sqrt{1+\lambda' n^2} + \frac{g_2^2\lambda'}{4\pi^2}|f|^2\left(\frac{1}{12}+\frac{35}{32\pi^2 n^2}\right)\left(\frac{3}{4}-\frac{n^2}{2}\lambda' + \mathcal{O}(\lambda'^2)\right) \tag{1.5}$$

where f is the unknown pre-factor of the vertex. Note that now, remarkably, if we set the pre-factor $|f|^2 = \frac{4}{3}$, the order λ' term agrees with gauge theory. The order λ'^2 term, however, does not. Further to this, the fractional powers of λ' are absent, at least up to order 7/2.

The essential new aspect of this computation is the use of the Dobashi-Yoneya vertex. Unlike the case of Minkowski space, on the pp-wave background there are competing proposals for the three string vertex. The original one [26, 35, 39, 46, 55] (which we will call the SVPS vertex) was fixed using the supersymmetry algebra up to a pre-factor function of the light-cone momentum (which is f in eq. (1.5)). Another vertex was proposed in ref. [53] and we will call it the DVPPRT vertex. The DVPPRT vertex solves the supersymmetry algebra in the simplest possible way, by acting upon the oscillator representation of the Dirac delta function which enforces world-sheet locality by the *quadratic* Hamiltonian and supercharge. This vertex is trivial in Minkowski space, but is non-trivial in the pp-wave background.

Then, in ref. [64], Dobashi and Yoneya proposed another form for the cubic Hamiltonian and supercharge based on consistency with the AdS/CFT holographic relations for three-point functions [2, 3] and their comparison with supergravity. This "holographic" vertex, which we shall call the DY vertex, is an equal-weighted average of the original SVPS vertex and the DVPPRT vertex: DY=$\frac{1}{2}$SVPS+$\frac{1}{2}$DVPPRT. It, and the four-string contact term that is generated using the supersymmetry algebra, are the vertices that are used in deriving eq. (1.5). A correction to the DY vertex based on the supersymmetry algebra was suggested in ref. [67]. Because of the truncation to the 2-impurity channel, this modification does not influence the computations in the present paper.

Though the result (1.5) is a big improvement on the previous one, it is still not in complete agreement with the gauge theory computation. It disagrees at order λ'^2. There might (or might not) be a simple reason for this disagreement. We have not performed computations beyond the 2-impurity channel. It could be that higher impurity channels contribute only to orders λ'^2 or higher, but do not influence the order λ' contribution. This would require a miraculous cancelation of a number of orders in the small $\lambda'^{\frac{1}{2}}$ expansion. After all, from power counting and the generic structure of the amplitude, one would expect that higher impurities begin to contribute at order $\lambda'^{\frac{1}{2}}$. In previous work, we have shown that this leading order cancels [68]. There, it was associated with cancelation of divergences, which were also generic, and supersymmetry played an important role. Examining whether this cancelation could also occur at orders λ' and $\lambda'^{\frac{3}{2}}$ is a challenge that has not been addressed yet. A careful check of this possibility would be very interesting.

There is another possibility for discrepancy. In all computations to date, the contact term with the supercharge $g_2^2 Q_4$ has been assumed to not contribute. Indeed, the super-

[2]There are a few minor corrections which affect the fractional powers in (1.4), but they remain non-zero in the corrected (1.4).

symmetry algebra shows that the contact term is (schematically) $g_2^2 H_4 = g_2^2\{Q_3, Q_3\} + g_2^2\{Q_2, Q_4\} + g_2^2\{Q_4, Q_2\}$ and only the first term on the right-hand side has been used in all computations. Generally, these contact terms are needed to cancel divergences arising from iterations of lower order vertices [70, 71]. In principle, Q_4 could be determined by finding multi-string matrix elements of the supersymmetry algebra. To our knowledge, this has not been attempted on the pp-wave background. We only observe that Q_4 is not needed to cancel divergences in any of the quantities that we compute. (This was also found on the Minkowski background in ref. [71].) It is easy to show that setting $Q_4 = 0$ is a solution of the supersymmetry algebra to the order that we are working. However, we cannot rule out its having a non-zero finite contribution that would affect our results.

One further observation that we can make is that, we could consider any linear combination of the SVPS and DVPPRT vertices: αSVPS+βDVPPRT. In this case, it would seem that, by using the supersymmetry algebra, one could consistently construct higher order contact terms in the Hamiltonian and supercharges, so that this is also a viable possibility for the vertex. In particular, we will see that divergences in the energy shifts of 2-impurity states cancel for any values of α and β. However, if we use this vertex to compute the energy shift, we find a result that agrees with gauge theory (1.3) to the leading order λ' only for the particular combination in the DY vertex, that is only when $\alpha = \beta = \frac{1}{2}$.

There is another intriguing and unexplained feature of these results, which was observed in ref. [49]. Consider the expansion of the string field theory Hamiltonian into free (quadratic) and interacting — cubic, quartic, etc. terms, $H = H_2 + g_2 H_3 + g_2^2 H_4 + \cdots$ and the expression for second order quantum mechanical perturbation theory which is used to compute (1.4),

$$\delta E^{(2)} \;=\; g_2^2 \langle\psi_0| H_3 \frac{1}{E_0 - H_2} H_3 |\psi_0\rangle + g_2^2 \langle\psi_0|H_4|\psi_0\rangle. \tag{1.6}$$

If, in the computation which arrives at (1.4), we change the terms on the right-hand-side of (1.6) by a relative factor of 2, either multiplying the first term by $\frac{1}{2}$ or the second term by 2, then the order λ' term would be different from that quoted in (1.4) and in that case the pre-factor could be chosen so that the order λ' term agrees with gauge theory. Here, we observe that this interesting fact persists in (1.4) to higher orders. In that case, with factor of 2 and the same choice of prefactor the order λ'^2 term also agrees with gauge theory, and the $\lambda'^{3/2}$ term vanishes. In addition, this intriguing fact persists in the computation of (1.5), if one inserts a relative factor of 2 in (1.6), (1.5) is modified so that it agrees with gauge theory up to and including order λ'^2 and the coefficients of the fractional powers with exponents 3/2 or 5/2 still vanish. At this point, we have no explanation for this fact. Inserting the factor of 2 is definitely not mathematically correct here. Aside from the violence it would do to quantum mechanical perturbation theory, it would upset the divergence cancelation that was found in ref. [68], for example. The reason, if any, for this numerological coincidence remains a mystery.

In the remainder of this Paper, we will outline the computation that leads to eq. (1.5). The notation and techniques are identical to those used in ref. [65] and ref. [68] and we defer to them for the details.

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2. Divergence cancelation

The light-cone energy of the two-oscillator free string state on the pp-wave background is $p^- = 2\mu\sqrt{1+\lambda' n^2}$. This matches the energy of the two impurity BMN operator in planar Yang-Mills theory. The energy-shift of these states due to string loop corrections is calculated in second order quantum mechanical perturbation theory using the formula in eq. (1.6). We will call the first term in (1.6) the "H_3 term" and the second the "contact term".

In our previous paper [68] we showed that, in the computation of the energy-shifts of some two-impurity states using SVPS vertex, the H_3 and contact terms individually contain logarithmically divergent sums over intermediate state mode numbers. These divergences were shown to always cancel, leaving a finite result which leads as $g_2^2\lambda'$. This behavior was shown to be generic, and to exist at arbitrary order in intermediate state impurities. This was important because, of course it is necessary to obtain finite amplitudes. In addition, it is also the mechanism whereby the leading order $\lambda'^{\frac{1}{2}}$ contributions cancel.

We shall now show that this mechanism is at play for the DVPPRT vertices, and that any linear combination of the SVPS and DVPPRT vertices will similarly be divergence free. The special choice of an equal weighted average - the DY vertex - is thus well behaved.

The simplest method to understand the divergence cancelation is to consider the energy shift of the two-impurity trace state

$$|[\mathbf{1},\mathbf{1}]\rangle = \frac{1}{2}\alpha_n^{i\dagger}\alpha_{-n}^{i\dagger}|\alpha\rangle \tag{2.1}$$

restricted to the impurity conserving channel. For details of this computation we refer the reader to [68] and for details of definitions and notation to ref. [65] and other literature quoted there. The DVPPRT vertex is given by the following expressions [53],

$$\begin{aligned}
|H_3^D\rangle &= -g_2\, f\left(\mu\alpha_3\,,\,\frac{\alpha_1}{\alpha_3}\right)\frac{\alpha'}{16\,\alpha_3^3}\Big[K^2+\widetilde{K}^2-4Y^{\alpha_1\alpha_2}\widetilde{Y}_{\alpha_1\alpha_2}-4Z^{\dot\alpha_1\dot\alpha_2}\widetilde{Z}_{\dot\alpha_1\dot\alpha_2}\Big]|V\rangle\,,\\
|Q^D_{3\,\beta_1\dot\beta_2}\rangle &= g_2\,\eta\, f\left(\mu\alpha_3\,,\,\frac{\alpha_1}{\alpha_3}\right)\frac{1}{4\,\alpha_3^3}\sqrt{-\frac{\alpha'\kappa}{2}}\Big(Z_{\dot\gamma_1\dot\beta_2}K^{\dot\gamma_1}_{\beta_1}-iY_{\beta_1\gamma_2}K^{\gamma_2}_{\dot\beta_2}\Big)|V\rangle\,,\\
|Q^D_{3\,\dot\beta_1\beta_2}\rangle &= g_2\,\bar\eta\, f\left(\mu\alpha_3\,,\,\frac{\alpha_1}{\alpha_3}\right)\frac{1}{4\,\alpha_3^3}\sqrt{-\frac{\alpha'\kappa}{2}}\Big(Y_{\gamma_1\beta_2}K^{\gamma_1}_{\dot\beta_1}-iZ_{\dot\beta_1\dot\gamma_2}K^{\dot\gamma_2}_{\beta_2}\Big)|V\rangle\,.
\end{aligned}\tag{2.2}$$

Unlike the SVPS case, the H_3 divergence does not stem from the two-bosonic-impurity intermediate state. This can be traced to the substitution of $K^2+\widetilde{K}^2$ for $K\,\widetilde{K}$ in the H_3 prefactor. There is, however, another divergence that was not present in the SVPS case. It is due to the contribution coming from matrix elements with two fermionic impurities in the intermediate state. In particular, the relevant matrix elements are given by

$$\begin{aligned}
&\langle\alpha_3|\alpha_n^i\alpha_{-n}^i\,\langle\alpha_2|\langle\alpha_1|\beta^{\alpha_1\alpha_2}_{p(1)}\beta_{-p(1)\beta_1\beta_2}|H_3^D\rangle =\\
&4\,g_2 r\,(1-r)\left(\frac{\omega_n^{(3)}}{\alpha_3}+\frac{\omega_p^{(1)}}{\alpha_1}\right)\left(\widetilde{Q}^{1\,1}_{-p\,p}\;-\widetilde{Q}^{1\,1}_{p\,-p}\right)\widetilde{N}^{3\,3}_{-n\,n}\delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2}_{\beta_2}
\end{aligned}\tag{2.3}$$

and similarly for the intermediate state with dotted indices. The divergent contribution to the energy shift coming from these matrix elements is found by taking the large p limits of the summands in (1.6). One finds

$$\delta E^{\rm div}_{H_3^D} \sim -\frac{1}{2}\int_0^1 dr\ \frac{g_2^2\, r(1-r)}{r\,|\alpha_3|\,\pi^2}\left(\widetilde{N}^{3\,3}_{n\,-n}\right)^2 \sum_p \frac{1}{|p|} \tag{2.4}$$

The contribution from the contact term stems from the following matrix element,

$$\begin{aligned}&\left(g_2\frac{\eta}{4}\sqrt{\frac{r\,(1-r)\,\alpha'}{-2\,\alpha_3^3}}\right)^{-1} \langle\alpha_3|\alpha^i_n\alpha^i_{-n}\,\langle\alpha_2|\langle\alpha_1|\alpha_p^{K\,(1)}\beta_{-p}^{(1)\,\Sigma_1\,\Sigma_2}|Q^D_{3\,\beta_1\dot\beta_2}\rangle = \\ &2\left(G^{(1)}_{|p|}\,K^{(3)}_{-n}\widetilde{N}^{3\,1}_{n\,p} + G^{(1)}_{|p|}\,K^{(3)}_{n}\widetilde{N}^{3\,1}_{-n\,p}\right)(\sigma^k)^{\dot\sigma_1}_{\beta_1}\delta^{\dot\sigma_2}_{\dot\beta_2} + 8\,G^{(1)}_{|p|}\,K^{(1)}_p\widetilde{N}^{3\,3}_{n\,-n}(\sigma^K)^{\Sigma}_{\beta}\delta^{\Sigma}_{\beta}.\end{aligned} \tag{2.5}$$

The divergent contribution to the energy shift is found to be,

$$\delta E^{\rm div}_{H_4^D} \sim +\int_0^1 dr\ \frac{g_2^2\, r(1-r)}{r\,|\alpha_3|\,\pi^2}\left(\widetilde{N}^{3\,3}_{n\,-n}\right)^2 \sum_{p>0} \frac{1}{p} \tag{2.6}$$

Noting that in the H_3^D contribution the divergence is found for both positive and negative p, while in the H_4^D contribution the divergence occurs only for negative p, and hence a relative factor of 2 is induced in the H_3^D term, one sees that the logarithmically divergent sums cancel identically between the H_3^D and contact terms, leaving a convergent sum. This result can be generalized to arbitrary impurity channels, as was done for the SVPS case in [68].

We now show that an arbitrary linear combination of the SVPS and DVPPRT vertices,

$$H_3^N = \alpha\, H_3^S + \beta\, H_3^D \tag{2.7}$$

$$Q_3^N = \alpha\, Q_3^S + \beta\, Q_3^D \tag{2.8}$$

similarly yields a finite energy shift. The divergence stemming from the H_3 term is simply α^2 times the SVPS H_3 divergence plus β^2 times (2.4). The reason is simple - the SVPS divergence stems from an entirely bosonic intermediate state, while (2.4) results from an entirely fermionic one. This precludes any divergences arising from cross terms. Referring the reader to equation (2.7) of [68], we note that the SVPS divergence is exactly equal to (2.4), therefore we have,

$$\delta E^{\rm div}_{H_3^N} \sim -(\alpha^2+\beta^2)\frac{1}{2}\int_0^1 dr\ \frac{g_2^2\, r(1-r)}{r\,|\alpha_3|\,\pi^2}\left(\widetilde{N}^{3\,3}_{n\,-n}\right)^2 \sum_p \frac{1}{|p|} \tag{2.9}$$

The pieces of the SVPS Q_3 relevant to a two-impurity channel calculation are exactly Q_3^D with $K \leftrightarrow \widetilde{K}$, see again [68].

$$\left(g_2\frac{\eta}{4}\sqrt{\frac{r\,(1-r)\,\alpha'}{-2\,\alpha_3^3}}\right)^{-1} \langle\alpha_3|\alpha^i_n\alpha^i_{-n}\,\langle\alpha_2|\langle\alpha_1|\alpha_p^{K\,(1)}\beta_{-p}^{(1)\,\Sigma_1\,\Sigma_2}|Q^Y_{3\,\beta_1\dot\beta_2}\rangle =$$

$$2G^{(1)}_{|p|}\left(\left[\alpha\,(K^{(3)}_{-n}\widetilde{N}^{3\,1}_{-n\,p}+K^{(3)}_{n}\widetilde{N}^{3\,1}_{n\,p})+\beta\,(K^{(3)}_{-n}\widetilde{N}^{3\,1}_{n\,p}+K^{(3)}_{n}\widetilde{N}^{3\,1}_{-n\,p})\right](\sigma^k)^{\dot\sigma_1}_{\dot\beta_1}\delta^{\dot\sigma_2}_{\dot\beta_2}\right.$$
$$\left.+\,4\,(\beta\,K^{(1)}_{p}+\alpha\,K^{(1)}_{-p})\widetilde{N}^{3\,3}_{n\,-n}(\sigma^K)^{\Sigma}_{\beta}\delta^{\Sigma}_{\beta}\right) \qquad (2.10)$$

The last term in (2.10) gives rise to a log-divergent sum, the large-p behaviour of which is:

$$\delta E^{\rm div}_{H_4^N}\sim+(\alpha^2+\beta^2)\int_0^1 dr\;\frac{g_2^2\,r(1-r)}{r\,|\alpha_3|\,\pi^2}\left(\widetilde{N}^{3\,3}_{n\,-n}\right)^2\sum_{p>0}\frac{1}{p} \qquad (2.11)$$

Thus the energy shift is finite for arbitrary α and β. The DY vertex uses $\alpha=\beta=1/2$, and this combination exclusively gives rise to the agreement with gauge theory discussed in the introduction. The generalization of these arguments to the impurity non-conserving channels is a straightforward application of the treatment given in [68].

3. Results

The calculations undertaken in this Paper are practically identical to those in [65], using the DVPPRT and DY vertices in place of the SVPS vertices used there. One small difference in the case of the SVPS vertex is that the half-integer powers of λ' calculated in ref. [65] and quoted vertabim in our eq. (1.4) are incomplete and suffer from a sign error, and are correctly given below. We refer the reader to this reference for details, and simply give results below.

The external state for which we are calculating the energy shift is

$$|[\mathbf{9},\mathbf{1}]\rangle^{(ij)}=\frac{1}{\sqrt{2}}\left(\alpha^{\dagger\,i}_{n}\alpha^{\dagger\,j}_{-n}+\alpha^{\dagger\,j}_{n}\alpha^{\dagger\,i}_{-n}-\frac{1}{2}\delta^{ij}\alpha^{\dagger\,k}_{n}\alpha^{\dagger\,k}_{-n}\right)|3\rangle.$$

For this particular state, individual H_3 and contact terms are not divergent in the two impurity approximation. It should be further noted that for this state, and for the impurity conserving channel, we shall find that use of the DY vertex, rather than the SVPS vertex, is equivalent to making the replacements of the quantities $(K,\widetilde{K})$ as $K\to(K+\widetilde{K})/2$ and $\widetilde{K}\to(K+\widetilde{K})/2$ in the SVPS vertex. This is the simplest way of reproducing our results.

The separate H_3 and contact term contributions to the energy shift for each of the three vertices are given below. We find that the DY energy shift agrees with gauge theory only at the leading order, while also enjoying the vanishing of the 3/2 and 5/2 powers of λ'. The order-λ'^2 term is of the correct form, but suffers from an overall factor of 4/3. The SVPS and DVPPRT results do not agree with gauge theory at the leading order. By multiplying the contact terms by two (an unjustified operation), one can recover the correct gauge theory result up to λ'^2 order with the SVPS (including vanishing of its $\lambda'^{3/2}$ term) and DY vertices. Further, this operation does not spoil the vanishing 3/2 and 5/2 powers of λ' for the DY result.

3.1 H_3 terms

$$\delta E^{\rm SVPS}_{H_3}=\frac{g_2^2}{32\pi^2}\left[\frac{15}{2\pi^2n^2}\lambda'+3\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right)\lambda'^{3/2}-\frac{27}{4\pi^2}\lambda'^2-n^2\left(\frac{5}{\pi^2}+\frac{9}{4\pi}\right)\lambda'^{5/2}\right.$$

$$+ \frac{111n^2}{16\pi^2}\lambda'^3 + n^4\left(\frac{45}{16\pi} + \frac{33}{5\pi^2}\right)\lambda'^{7/2} + \mathcal{O}(\lambda'^4)\Bigg] \tag{3.1}$$

$$\begin{aligned} \delta E_{H_3}^{\rm DVPPRT} &= \frac{g_2^2}{32\pi^2}\left[-\left(\frac{2}{3} + \frac{5}{4\pi^2 n^2}\right)\lambda' + 3\left(\frac{1}{\pi^2} + \frac{1}{2\pi}\right)\lambda'^{3/2} + n^2\left(1 - \frac{9}{8\pi^2 n^2}\right)\lambda'^2\right. \\ &- 5n^2\left(\frac{2}{\pi^2} + \frac{3}{4\pi}\right)\lambda'^{5/2} - 5n^4\left(\frac{1}{4} - \frac{21}{32\pi^2 n^2}\right)\lambda'^3 + n^4\left(\frac{105}{16\pi} + \frac{94}{5\pi^2}\right)\lambda'^{7/2} \\ &+ \left.\mathcal{O}(\lambda'^4)\right] \end{aligned} \tag{3.2}$$

$$\begin{aligned} \delta E_{H_3}^{\rm DY} &= \frac{g_2^2}{4\pi^2}\left[\frac{3}{4}\left(\frac{1}{12} + \frac{35}{32\pi^2 n^2}\right)\lambda' - 5n^2\left(\frac{1}{96} + \frac{35}{256\pi^2 n^2}\right)\lambda'^2\right. \\ &+ \left. n^4\left(\frac{17}{384} + \frac{655}{1024\pi^2 n^2}\right)\lambda'^3 + n^4\left(\frac{3}{256\pi} + \frac{23}{640\pi^2}\right)\lambda'^{7/2} + \mathcal{O}(\lambda'^4)\right] \end{aligned} \tag{3.3}$$

3.2 Contact terms

$$\begin{aligned} \delta E_{H_4}^{\rm SVPS} &= \frac{g_2^2}{32\pi^2}\left[\left(\frac{1}{3} + \frac{5}{8\pi^2 n^2}\right)\lambda' - \frac{3}{2}\left(\frac{1}{\pi^2} + \frac{1}{2\pi}\right)\lambda'^{3/2} - n^2\left(\frac{1}{6} - \frac{19}{16\pi^2 n^2}\right)\lambda'^2\right. \\ &+ n^2\left(\frac{11}{4\pi^2} + \frac{9}{8\pi}\right)\lambda'^{5/2} + \frac{n^4}{8}\left(1 - \frac{105}{8\pi^2 n^2}\right)\lambda'^3 - n^4\left(\frac{45}{32\pi} + \frac{73}{20\pi^2}\right)\lambda'^{7/2} \\ &+ \left.\mathcal{O}(\lambda'^4)\right] \end{aligned} \tag{3.4}$$

$$\delta E_{H_4}^{\rm DVPPRT} = \delta E_{H_4}^{\rm SVPS} \tag{3.5}$$

$$\begin{aligned} \delta E_{H_4}^{\rm DY} &= \frac{g_2^2}{4\pi^2}\left[n^2\left(\frac{1}{96} + \frac{35}{256\pi^2 n^2}\right)\lambda'^2 - \frac{5n^4}{128}\left(\frac{1}{3} + \frac{29}{8\pi^2 n^2}\right)\lambda'^3\right. \\ &+ \left.\frac{n^4}{256}\left(\frac{3}{2\pi} + \frac{5}{\pi^2}\right)\lambda'^{7/2} + \mathcal{O}(\lambda'^4)\right] \end{aligned} \tag{3.6}$$

3.3 Energy shifts

The results for the complete energy shifts are as follows,

$$\begin{aligned} \delta E^{\rm SVPS} &= \frac{g_2^2}{4\pi^2}\left[\left(\frac{1}{24} + \frac{65}{64\pi^2 n^2}\right)\lambda' + \frac{3}{16}\left(\frac{1}{\pi^2} + \frac{1}{2\pi}\right)\lambda'^{3/2}\right. \\ &- n^2\left(\frac{1}{48} + \frac{89}{128\pi^2 n^2}\right)\lambda'^2 - \frac{9\,n^2}{32}\left(\frac{1}{\pi^2} + \frac{1}{2\pi}\right)\lambda'^{5/2} \\ &+ \left. n^4\left(\frac{1}{64} + \frac{339}{512\pi^2 n^2}\right)\lambda'^3 + n^4\left(\frac{59}{160\pi^2} + \frac{45}{256\pi}\right)\lambda'^{7/2} + \mathcal{O}(\lambda'^4)\right] \end{aligned} \tag{3.7}$$

$$\delta E^{\rm DVPPRT} = \frac{g_2^2}{4\pi^2}\left[-\left(\frac{1}{24} + \frac{5}{64\pi^2 n^2}\right)\lambda' + \frac{3}{16}\left(\frac{1}{\pi^2} + \frac{1}{2\pi}\right)\lambda'^{3/2}\right.$$

$$+ n^2\left(\frac{5}{48}+\frac{1}{128\pi^2 n^2}\right)\lambda'^2 - n^2\left(\frac{29}{32\pi^2}+\frac{21}{64\pi}\right)\lambda'^{5/2}$$
$$+ n^4\left(-\frac{9}{64}+\frac{105}{512\pi^2 n^2}\right)\lambda'^3 + n^4\left(\frac{303}{160\pi^2}+\frac{165}{256\pi}\right)\lambda'^{7/2}+\mathcal{O}(\lambda'^4)\Bigg] \quad (3.8)$$

$$\delta E^{\rm DY} = \frac{g_2^2}{4\pi^2}\frac{3}{4}\left[\left(\frac{1}{12}+\frac{35}{32\pi^2 n^2}\right)\left(\lambda' - \frac{4}{3}\frac{n^2}{2}\lambda'^2\right) + \frac{n^4}{24}\left(1+\frac{255}{16\pi^2 n^2}\right)\lambda'^3 + \frac{n^4}{384}\left(\frac{9}{\pi}+\frac{142}{5\pi^2}\right)\lambda'^{7/2}+\mathcal{O}(\lambda'^4)\right] \quad (3.9)$$

Recall that the leading 3/4 is irrelevant and can be scaled away by fixing the overall f factor which multiplies the vertices (and which has not been written in the above formulae, where it would appear in each as an overall factor of $|f|^2$). We see that the gauge theory result (1.3) is matched only by the DY result, and only at leading order in λ', with the λ'^2 term being of the correct form but with an overall factor of 4/3. We also see the miraculous absence of the $\lambda'^{3/2}$ and $\lambda'^{5/2}$ terms which are clearly generic in the string field theory. The result (3.9) represents the best matching of this quantity to gauge theory so far, and thus is an indication that the DY vertex is an improvement over its predecessors.

Mysteriously, if the contact terms are scaled by a factor of 2, the agreement with gauge theory is enhanced for both the SVPS and DY results,

$$\delta E^{\rm SVPS}_{2H_4} = \frac{g_2^2}{4\pi^2}\left[\left(\frac{1}{12}+\frac{35}{32\pi^2 n^2}\right)\left(\lambda' - \frac{n^2}{2}\lambda'^2\right) + \frac{n^2}{16\pi^2}\lambda'^{5/2} + n^4\left(\frac{1}{32}+\frac{117}{256\pi^2 n^2}\right)\lambda'^3 - \frac{7n^4}{80\pi^2}\lambda'^{7/2}+\mathcal{O}(\lambda'^4)\right] \quad (3.10)$$

$$\delta E^{\rm DY}_{2H_4} = \frac{g_2^2}{4\pi^2}\frac{3}{4}\left[\left(\frac{1}{12}+\frac{35}{32\pi^2 n^2}\right)\left(\lambda' - \frac{n^2}{2}\lambda'^2\right) + n^4\left(\frac{7}{288}+\frac{365}{768\pi^2 n^2}\right)\lambda'^3 + n^4\left(\frac{1}{10\pi^2}+\frac{1}{32\pi}\right)\lambda'^{7/2}+\mathcal{O}(\lambda'^4)\right] \quad (3.11)$$

however, the DY result is still superior in that the $\lambda'^{5/2}$ power is absent.

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